Filosofia Unisinos
Unisinos Journal of Philosophy
17(3):248-262, sep/dec 2016
Unisinos – doi: 10.4013/fsu.2016.173.01

The assessment of changing in mathematical ends in Maddy's philosophy

A avaliação de mudanças nos objetivos da matemática na filosofia de Maddy

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ABSTRACT

In Maddy's philosophy, mathematics is autonomous, i.e., it is not subordinated to either science or philosophy. Mathematics establishes and pursues its own goals and must be judged on its own terms. This leads Maddy to admit, in *Naturalism in Mathematics* (1997) and also in *Second Philosophy* (2007), that, even if mathematicians choose to pursue a goal that could seem improper from the philosophical or scientific point of view, there is nothing to be done except accepting the new state of affairs. In *Defending the Axioms* (2011), nevertheless, Maddy changes her position regarding this issue. She claims to have found the basis from which to assess the adequacy of mathematical goals. From this basis, if mathematicians choose to pursue what seems to be an improper goal, the philosopher could claim that they are going astray. In this paper, I will review Maddy's positions in these books; and, especially regarding *Defending the Axioms*, I will sustain that the institution of a permanent parameter for the judgment of mathematical goals goes against the alleged autonomy of mathematics and other important traits of her philosophy.

Keywords: Penelope Maddy, autonomy of mathematics, mathematical ends.

RESUMO

Na filosofia de Maddy, a matemática é autônoma, isto é, não está subordinada nem à ciência nem à filosofia. A matemática estabelece e persegue suas próprias metas e deve ser julgada em seus próprios termos. Isso faz com que Maddy admita, em *Naturalism in Mathematics* (1997) e também em *Second Philosophy* (2007), que, ainda que os matemáticos decidam perseguir uma meta que pareça imprópria do ponto de vista filosófico ou científico, não há nada a ser feito a não ser aceitá-la. Em *Defending the Axioms* (2011), no entanto, Maddy muda sua posição sobre esse ponto e afirma ter encontrado bases para julgar a adequação das metas da matemática. A partir dessas bases, se os matemáticos decidirem buscar metas que se pareçam impróprias, a filósofa poderia dizer que eles estão seguindo o caminho errado. Neste artigo, recapitularemos as posições de Maddy nessas obras. Em especial sobre sua posição em *Defending the Axioms,* sustentaremos que a instituição de um parâmetro permanente para o julgamento das metas da matemáticas vai contra a alegação de sua autonomia e outros traços importantes da filosofia de Maddy.

Palavras-chave: Penelope Maddy, autonomia da matemática, objetivos da matemática.

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Introduction

Penelope Maddy has conducted an important study on the methodology of mathematics from a naturalistic point of view. The questions of what the proper methods of mathematics are and what justifies them are among the main concerns of her philosophy. Her strategy for answering these questions is grounded on an analysis of real episodes in mathematical practice. In the naturalistic spirit, Maddy subscribes the idea that the facts of the practice must precede any theory about them. Indeed, from Maddy's perspective, mathematical practice, as it is performed by real mathematicians, is the parameter for assessing the appropriateness of any philosophical account of the methods of mathematics. Thus, in a regular scientific way, from the careful examination of mathematical practice, Maddy intends to develop a faithful account of mathematics and its methods. On the same basis, she criticizes other philosophical theses on mathematics—such as Quine's indispensability argument, as explained below—by showing how distant they are from real mathematical practice.

Nevertheless, Maddy's philosophy of mathematics, even if firmly based on mathematical practice, might have its own shortcomings. Mathematics, as it has been undertaken by past and present mathematicians, changes. The history of mathematics is there to show this. Consequently, a philosophical thesis or even a mathematical method that was appropriate in the past can be inappropriate now, and vice versa. In fact, this is a common source of trouble for many naturalistic methodologies, both in philosophy of science and philosophy of mathematics. Feferman addresses this issue regarding Maddy's naturalism:

> *While Maddy keeps invoking mathematical practice in general in the scope of her naturalism, she does not reflect on the many instances in its history in which the question of what entities are to be admitted to mathematics and what methods are legitimate had to be faced, leading to substantial revisions from what's OK to what's not OK and vice versa. In binding itself to mathematical practice, this kind of naturalism is in danger of being unduly transitory. Even if one takes the proposed naturalistic point of view and mathematical practice as exemplified in set theory for granted, there is a crucial question as to what determines the "mathematical ends" for which the "most effective" means are to be sought (Feferman* et al.*, 2000, p. 409).*

Here, Feferman criticizes Maddy naturalistic strategy from a traditional perspective. Traditional methodological studies tend to see science or mathematics as having a permanent, ultimate aim, such as seeking the truth about a certain realm of entities. Given that these aims are immutable, the methods of inquiry are also immutable. Thus, from the traditional perspective, "the scientific method" must be the appropriate method for all scientists and must have been so throughout history. The same would be the case concerning mathematics. There must be "the mathematical method" that is established once and for all. Naturalized methodologies, on the other hand, tend to relativize aims to a certain epoch and community. From the latter perspective, since the aims of science and mathematics are not set up once and for all, the most appropriate methods of inquiry can vary over time and within different research communities. However, without a permanent, ultimate end, what determines these transitory goals?

This relativization of aims and methods is a well-known trait of Kuhn's and Feyerabend's philosophies of science. Given that they prioritize the analysis of episodes in real scientific practice, and given that scientific practices change over time, they unreservedly accept the transitoriness of aims and methods as fact. Ultimately, according to them, it is the scientific community who establishes its own aims.

In Maddy's philosophy of mathematics, however, this is not taken for granted. Although Maddy also prioritizes the analysis of real mathematical practice and mathematical practice also changes over time, in her most recent works Maddy has resisted embracing the idea that it is the mathematical community who establishes its own aims. In fact, Maddy has changed her position regarding this issue over the years. In *Naturalism in Mathematics* (1997; henceforth NM), Maddy admitted as inevitable the conclusion that, if the entire mathematical community decides to pursue a new goal, the naturalist philosopher can do nothing but accept it. She regards the transitoriness of which Feferman accuses her as a positive feature of her philosophy. In *Second Philosophy* (2007; henceforth SP), however, Maddy envisages an alternative to the Second Philosopher. The Second Philosopher is a naturalist and yet prioritizes real mathematical practice as well, but she claims that, if the entire mathematical community happens to pursue a new goal that is improper from her point of view, she can put aside the mathematical community and find a new field of research that will seek the goals she regards as appropriate. In *Defending the Axioms* (2011; henceforth DA), Maddy adopts a more radical attitude. If the mathematical community embarks on the pursuit of a goal that is unacceptable, the Second Philosopher can criticize the mathematical community and declare that they are going astray. In DA, she claims to have eventually found the grounds on which to determine the mathematical ends for which Feferman asks.

In this paper, I intend to review the different positions Maddy assumes in NM, SP, and DA regarding changes in mathematical goals, and also assess the extent to which her answers to the question of what determines mathematical goals are coherent with her assumed philosophical background. Specifically, I want to show that her position in DA is incompatible with most of the Second Philosopher's thinking. In order to fulfil this aim, let me start with a brief review of Maddy's naturalism in mathematics.

Maddy's naturalism

Maddy's naturalism is an heir of Quine's naturalism. In NM, Maddy presents her naturalism in mathematics as characterized by a concession to mathematics of the same rights that Quine's naturalism concedes to the natural sciences.

Concerning the natural sciences, Quine states that "it is within science itself, and not in some prior philosophy, that reality is to be identified and described" (Quine, 1981a, p. 21). According to him, the best theories and the best methods of inquiry into reality are scientific ones. Traditional philosophical theories and methods, in Quine's opinion, are not as successful as scientific ones. As a result, there is no point in traditional attempts at grounding scientific knowledge and methods on pure a priori philosophical reasons—i.e., first philosophy—, since the theses and methods of first philosophy are clearly more fragile and less reliable than those of the sciences. The core of Quine's naturalism is the assumption that science is an autonomous enterprise that is not in need of any extra-scientific justification. Maddy's naturalism follows Quine in this matter.

Nevertheless, Maddy and Quine do not have the same opinion regarding mathematics. For Quine, mathematics is not an autonomous enterprise; it is subordinated to the natural sciences. Mathematics deserves its status as 'knowledge,' according to Quine, only because it is useful to the natural sciences. The indispensability argument is the ontological face of this subordination. Briefly, the indispensability argument affirms that we are committed to the existence of mathematical entities to the same degree that we are committed to scientific theories where these entities are indispensably employed. For example, since the use of numbers by the best physical theories is indispensable, we must admit that numbers exist (cf. Quine, 1963). The existence of a certain mathematical entity depends on its application in the sciences. Mathematical entities that are not employed in the sciences, and are not extensions, in a certain sense, of those that are scientifically useful, do not have ontological rights, according to Quine. He sees such entities "only as mathematical recreation" (Quine, 1998, p. 400).

Maddy rejects Quine's indispensability argument and, consequently, the subordination of mathematics to sciences. Her rejection of these parts of Quinean naturalism is based on her historical analyses of mathematical practice.

> *Histories of nineteenth-century mathematics tell a compelling story of how mathematics gradually separated itself from physical science and undertook pursuits of its own—motivated by its own goals and interests, as well as those of science—but the Quinean naturalist persists in subordinating mathematics to science, on identifying the proper methods of mathematics with the methods of science (Maddy, 1997, NM, p. 184).*

Maddy points out that, at least since the nineteenth century, mathematicians have not looked to scientific results in order to solve ontological issues in their mathematical theories. If the indispensability argument is right, mathematicians concerned with, for example, independent questions in set theory, such as the continuum hypothesis, would be highly interested in possible scientific indispensable applications of the continuum. However, "set theorists are not attentive to these matters" (Maddy, 1997, NM, p. 158). Existential questions in mathematics are generally solved by pure mathematical considerations, which are independent of attention to applicability in the natural sciences. "My guess is that the practice of set theory, the methods set theorists actually use to pursue the independent questions, would be unaffected, no matter how these issues in natural science might turn out", Maddy concludes (1997, NM, p. 159). Contrary to Quine, Maddy states that contemporary pure mathematics is neither ontologically nor methodologically subordinate to the natural sciences.

The same is true regarding the relationship between mathematics and philosophy, according to Maddy. Her analysis of mathematical practice shows that philosophical considerations do not play any important role in mathematical issues, even if these issues are fundamental ones, such as the selection of axioms in set theory. This is contrary to the common understanding, which sustains that, although mathematical investigation in general can be seen as no more than deduction from the axioms of set theory—and, therefore, independent of philosophical issues—, the adoption of the fundamental axioms of set theory itself demands philosophical consideration. Maddy, rather, sustains that the main reasons that really justified the adoption of the axioms of standard set theory were entirely mathematical. She grounds this in her analysis of the history of set theory. The case of the Axiom of Infinity is exemplary. This axiom postulates the existence of a completed infinite set. Philosophically, the existence of completed infinite sets was and continues to be divisive. Aristotle and, more recently, renowned mathematicians such as Poincaré and Brouwer, accepted only the existence of potential infinite sets. If the adoption of the Axiom of Infinity in set theory had been philosophically motivated, one would expect the philosophical question regarding the existence of completed infinities to have been settled. Since this is not the case—the philosophical question remains open—, the necessary conclusion is that the reasons that supported the Axiom of Infinity were not philosophical. As a matter of fact, the main reason for the acceptance of the Axiom of Infinity was purely mathematical: without it, it is not possible to define real numbers inside set theory (cf. Maddy, 1997, NM, p. 51-52). In the face of its mathematical importance, philosophical concern regarding completed infinity is peripheral.

Similar reasoning is applied for each of the other set-theoretical axioms. Although philosophical issues have been raised in discussions about each of them—the debate about the Axiom of Choice is another good example—, the axioms were eventually accepted when mathematicians realized that they were necessary for proving important mathematical results, even if the related philosophical questions remained without answer. In sum, if a mathematical question was settled while the related philosophical question remained open, the necessary conclusion is that the latter is irrelevant to the former. Maddy's conclusion is as follows: "if you want to answer a question of mathematical methodology, look not to traditionally philosophical matters about the nature of mathematical entities, but to the needs and goals of mathematics itself" (Maddy, 1997, NM, p. 191).

Accordingly, mathematics is not only independent from the natural sciences, but also from philosophy. This defence of the autonomy of mathematics is the main point of Maddy's naturalism.

> *What I propose here is a mathematical naturalism that extends the same respect to mathematical practice that the Quinean naturalist extends to scientific practice. It is, after all, those methods—the actual methods of mathematics—not the Quinean replacements, that have led to the remarkable successes of modern mathematics. Where Quine holds that science is 'not answerable to any supra-scientific tribunal, and not in need of any justification beyond observation and the hypothetico-deductive method' …, the mathematical naturalist adds that mathematics is not answerable to any extra-mathematical tribunal and not in need of any justification beyond proof and the axiomatic method. Where Quine takes science to be independent of first philosophy, my naturalist takes mathematics to be independent of both first philosophy and natural science (including the naturalized philosophy that is continuous with science)—in short, from any external standard (Maddy, 1997, NM, p. 184).*

With mathematics no longer regarded as subordinated to the natural sciences, one might ask why Maddy insists on classifying her position as naturalist. From Maddy's perspective, however, the fact is that a responsible naturalist must admit the autonomy of mathematics.

> *To judge mathematical methods from any vantage-point outside mathematics, say from the vantage-point of physics, seems to me to run counter to the fundamental spirit that underlies all naturalism: the conviction that a successful enterprise, be it science or mathematics, should be understood and evaluated on its own terms, that such an enterprise should not be subject to criticism from, and does not stand in need of support from, some external, supposedly higher point of view (Maddy, 1997, NM, p. 184).*

In order not to betray what she considers to be the keystone of Quinean naturalism—the recognition of the autonomy of a successful enterprise—Maddy rejects the subordination of mathematics to the natural sciences. Furthermore, it is important to note that the autonomy of mathematics, in Maddy's naturalism, is not a matter of principle, but a consequence of a careful examination of significant episodes in real mathematical practice, such as the history of set theory. This examination of mathematical practice "takes place in a more-or-less sociological spirit", as she explains (Maddy, 1997, NM, p. 199). Accordingly, Maddy claims that mathematics is autonomous because, in fact, a sociological-like study of what mathematicians really do—her analysis of contemporary and historical mathematical practice—can show that mathematics is autonomous. Her analysis also demonstrates that the success of mathematics comes from its autonomy. To base philosophical conclusions on a careful empirical study of a phenomenon is a characteristic trait of naturalism. "[I]f our philosophical account of mathematics comes into conflict with successful mathematical practice, it is the philosophy that must give", she (Maddy, 1997, NM, p. 161) summarizes.

Even so, someone might argue against Maddy that, since many mathematicians have made philosophical claims in order to sustain their mathematical theses, resorting to philosophical reflexion could be regarded as a legitimate part of mathematical practice. Gödel can be mentioned as an example. His Platonist convictions are widely known, as are the use he makes of them in important mathematical issues. In the following excerpt, for instance, Gödel resorts to Platonism in order to reject proof of the independence of the continuum hypothesis as a final answer to Cantor's conjecture.

> *[T]he set-theoretical concepts and theorems describe some well-determined reality, in what Cantor's conjecture must be either true or false. Hence its undecidability from the axioms being assumed today can only mean that these axioms do not contain a complete description of that reality (Gödel, 1983, p. 476).*

In response to this potential objection, Maddy reminds us that one must distinguish between what mathematicians say and what they really do. In her opinion, evocations of philosophical beliefs by mathematicians should be seen as "colorful asides or heuristic aides, but not as part of the evidential structure of the subject" (Maddy, 2011, DA, p. 53). In NM (Maddy, 1997, p. 166), she addresses this issue by reiterating a distinction she takes from Wittgenstein between *prose* and *calculus*. She holds that philosophical talk on the part of mathematicians is mere *prose*, without determinate influence over mathematical matters. What is really decisive in mathematical debates is the *calculus*, that is, genuine mathematical motivations. Again, Gödel is a good example. Just after stating his Platonist convictions in the excerpt quoted above, Gödel evokes genuine mathematical considerations in order to sustain his position. He mentions the iterative concept of a set and the assumption of large cardinal axioms as a means of extending set theory without arbitrariness and with positive mathematical consequences (cf. Gödel, 1983, p. 477). The distinction between *prose* and *calculus* is clear. A mathematician who wanted to counter Gödel would better succeed by pointing out failures or fragilities in Gödel proposal to extending set theory than by criticizing his Platonist beliefs. The philosophical debate on Platonism is irrelevant in this case, since the mathematical force of Gödel's argument remains the same even after the philosophical asides are extirpated. Maddy (1997, NM, p. 175) concludes: "the mathematics, not the philosophy, is doing the work of establishing the legitimacy of the independent question".

The primacy Maddy concedes to what mathematicians do, in comparison with what they say, is far from being an artifice. She claims that this is good scientific practice. "[W]orking sociologists certainly allow that the testimony of subjects can be less than fully trustworthy, that accurate description of a practice sometimes requires a discounting of some participants' reports" (Maddy, 1997, NM, p. 199).

As we have seen, given the protagonism Maddy concedes to mathematical practice, her opinion in NM is that philosophy is neither in the position of recommending reform to mathematical methods nor in the position of justifying mathematical knowledge. In accordance with its Quinean roots, Maddy's naturalism in mathematics is descriptive and strictly explanatory. It does not aim to justify or ground mathematics in any sense. However, it does aim to contribute to contemporary mathematical debates. Maddy's naturalism can do this since a naturalized analysis of mathematical practice can reveal the methods that have really contributed to mathematical success on past occasions; and distinguish them from those that constituted mere distractions from the real issues, such as philosophical concerns. These effective methods, then, can be applied in order to help to settle contemporary open questions in mathematics. When doing this, Maddy affirms that the philosopher stops doing philosophy and enters mathematics itself.

> *At this point, the naturalist is not doing sociology or natural science of any kind; she is using the methods of mathematics, not those of science, and she is doing so exactly as a mathematician might, except that her choices among the available styles of argument are guided by the results of the previous historical analysis. In other words, she is functioning within mathematics, just as a mathematician might, except that she uses only those styles of argument that her previous analysis suggests are the relevant, effective ones. […] At this stage, the naturalist is doing what the sociologist might call 'going native' (Maddy, 1997, NM, p. 199).*

Indeed, this is one of the main purposes of NM, where Maddy elaborates a naturalistic argument against the Axiom of Constructibility, which is connected to the debate around the continuum hypothesis in set theory. The cornerstone of her argument is the concept of maximizing, which she defines in set theory itself (Maddy, 1997, NM, p. 216 ff). Her argument is intended to be mathematical, not philosophical. The naturalist aims to contribute to the mathematical debate by applying the most successful mathematical methods like a "native", that is, in the same way a working mathematician would do. A distinctive trace of the naturalist—which differentiates her from an ordinary mathematician—is her awareness about what the most effective methods are, acquired in her sociological and historical analysis of the practice. The selection of the most effective methods, however, is not based on purely sociological or historical consideration.

> *It is important to notice that the arguments [for or against a method] themselves are not sociological; the naturalist does not argue 'this method is preferable because it conforms to previous practice', but 'this method is preferable because it is the most effective method available for achieving this goal' (Maddy, 1997, NM, p. 199).*

The most successful methods are not necessarily those that have been employed more often in the past. This is neither a statistical nor a historical matter. The reference to a goal in the above excerpt is crucial. The sociological-like analysis of the practice is essential only in the stage of identification of goals and methods in the area. Once the goals are identified, the appropriateness of the methods are assessed regarding their effectivity in promoting the goals. This assessment is made following conventional means–ends reasoning. Maddy explains that "here the rationality of mathematics is being assessed in terms of some minimal theory of the rationality", which is not "external to mathematics: it is used in mathematics as it is in all human undertakings" (Maddy, 1997, NM, p. 197, footnote 8). The preferences of the community are no longer important. On this point, Maddy's naturalistic methodology departs from a Kuhnian-like naturalistic methodology. From Kuhn's perspective, the scientific community has the final word. The philosophical assessment of a certain method cannot disagree with the opinion of the scientific community. But, as Maddy assumes in advanced conventional means–ends reasoning standards, she can go against the preferences of the mathematical community.

Maddy's case against the Axiom of Constructibility provides an example of this. Her analysis of the history of set theory results in the conclusion that the main goal of most of the mathematicians who developed set theory was "to provide a single system in which all objects and structures of mathematics can be modelled or instantiated" (Maddy, 1997, NM, p. 208-209). This general goal leads to two more specific goals: unifying and maximizing. The unification—that is, the development of one standard set theory, rather than the profusion of a multitude of alternative set theories—is required since a single system is the aim. The maximization—that is, the amplification of the theory's power to represent the full diversity of mathematical objects—is required, roughly, because the theory aims to embrace all of mathematics. Given those two goals, what could the mathematician's be attitude towards the Axiom of Constructibility? Maddy advises set theorists in this way: if you want to both unify and maximize, you should reject the Axiom of Constructibility, because it is restrictive and jeopardizes the aim of maximization. It must be said that this advice is given in proper mathematical language and method, which means that Maddy defines mathematically, in set theory, the concepts of maximization and restrictiveness and shows, in set theory, how the Axiom of Constructibility is restrictive.

Maddy would give this advice even if most of the mathematical community were sympathetic to the Axiom of Constructibility. Her argument is not stronger because it coincides with the preferences of many set-theorists. Her claim is based on the rational judgment of the appropriateness of the mean—adoption of the Axiom of Constructibility—to the ends—unification and maximization. From this sole reasoning she concludes that the adoption of this axiom would be detrimental to the goals of set theory. This fact is independent of any preferences of the mathematical community.

Thus, although Maddy's naturalism is essentially descriptive, in the sense that it wants to neither criticize nor support mathematics, it can *advise* the mathematicians. After the naturalist has identified the goals and methods of a certain area of mathematics—paying attention to the crucial distinction between *prose* and *calculus*—, she can counsel mathematicians, saying things like: "if your goal is x, then you should proceed this way, since this is more likely to promote x than other alternatives". Such advice must be given in the appropriate mathematical language, and must be supported by the appropriate mathematical reasons, since the naturalized analysis of the practice has shown that extra-mathematical considerations are irrelevant to mathematical issues. At this stage, her argument is no longer philosophical or sociological, it is mathematical. Given the autonomy of mathematics, "going native" is the most effective way in which the naturalist can contribute to mathematics.

The naturalist attitude towards changes in goals

Given this description of Maddy's naturalism, what would her attitude be towards changes in the goals of mathematics? One of the main points of her naturalism provides a straightforward answer. She claims that mathematics is an autonomous enterprise, not subordinate either to the natural sciences or to philosophy. As long as there is no external ground from which mathematical methodological decisions concerning its goals can be judged, we must conclude that mathematics has the autonomy to establish its own goals. In fact, this is Maddy's opinion in NM: "the naturalist has no independent grounds on which to criticize or defend the actual goals of the practice" (Maddy, 1997, NM, p. 198).

She admits that mathematicians can be deluded about the goals they are effectively pursuing. In such a context, the naturalist analysis should carefully distinguish between what mathematicians say and what they really do. This would be just another case of *prose* versus *calculus*. "This is the type of error a sociologist might find in any human practice: by careful analysis, we see that the practice is directed towards achieving goals A and B, while practitioners give lip service to goal C. Nothing mysterious here" (Maddy, 1997, NM, p. 198). However, once the sociological-like analysis of the naturalist has found the actual goal of the practice, there is no choice but to accept it. "[T]he possibility of practitioner error is in the identification of goals, not in the choice of goals", Maddy (1997, NM, p. 198) concludes.

In order to reinforce this consequence of her naturalistic convictions, Maddy imagines what she calls a "wild example":

> *[S]uppose mathematicians decided to reject the old maxim against inconsistency—so that both '2 + 2 = 4' and '2 + 2 = 5' could be accepted—on the grounds that this would have a sociological benefit for the self-esteem of school children. This would seem a blatant invasion of mathematics by non-mathematical considerations, but if mathematicians themselves insisted that this was not so, that they were pursuing a legitimate mathematical goal, that this goal overrides the various traditional goals, I find nothing in the mathematical naturalism presented here that provides grounds for protest (Maddy, 1997, NM, p. 198, footnote 9).*

Even if the mathematical community were to choose to pursue a goal that, from the current perspective, seemed plainly improper, the naturalist would have no choice but to accept the new goal. It is worth remembering that, according Maddy's naturalism, if a philosophical account of mathematics comes into conflict with mathematical practice, it is the philosophy that must give (cf. Maddy, 1997, NM, p. 161). Therefore, if the naturalist arrives at a conclusion that is not supported by an analysis of the actual practice—such as would be the conclusion against the rejection of the consistency maxim in the above hypothetical case—this simply means that the naturalist's conclusion is wrong.

The adherence to conventional means-ends reasoning standards does not help in this case. The naturalist can criticize the methodological preferences of the community only regarding the goals the community has instituted for itself. The naturalist can advise mathematicians that, given the goals of the practice, one or another methodological attitude is improper, as we have seen above. However, on the definition of the goals, the naturalist has nothing to say.

In this way, Maddy's naturalism is unavoidably transitory. In a direct answer to Feferman's accusation, quoted above, Maddy recognizes this, but she refuses to accept that such a transitoriness is "undue":

> *Of course it is true that our rational judgment of which mathematical principles and methods are best will change as we learn more: we have less reason now to embrace infinitesimals than we did before Cauchy and Weierstrass; we have more reason now to embrace large cardinals than we did, for example, when Ulam first defined measurables in 1930. I don't see why this is any more alarming than the fact that scientists have more reason now to believe in atoms than they did before Einstein and Perrin; we learn new things, acquire new evidence, modify our theories, in both mathematics and science. So naturalistic justifications will shift as our understanding increases, but I don't think this makes those justifications any more 'unduly transitory' than our scientific theories (Maddy* in *Feferman* et al., *2000, p. 420-421).*

Rather than undue, Maddy regards the transitoriness of her naturalism as advantageous. There is nothing wrong with transitoriness, since our position changes in response to what we have learned over time. It is completely rational. On the contrary, continuing to hold a philosophical position no longer supported by successful mathematical practice would be irrational. In this sense, transitoriness is a positive consequence of a philosophy grounded on mathematical practice.

Second Philosophy

This was Maddy's position in NM. In SP, she keeps almost the same position, but she envisions an alternative course of action. In order to appreciate this, at first we have to understand some modifications to Maddy's naturalism introduced in SP. Briefly, *Second Philosophy* is an improved version of the naturalism of NM. "*Second Philosophy* ([2007]) lays out the broader philosophical background that seemed to me necessary", Maddy (2007, SP, p. ix) explains, to carry on her methodological study of pure mathematics in general and set theory in particular.

The most noticeable alteration from Maddy's previous position concerns the term she uses to refer to it. She relinquished the name naturalism: "the term 'naturalism' has acquired so many associations over the years that using it tends to invite indignant responses of the form, 'but that can't be naturalism! Naturalism has to be like this!'" (Maddy, 2007, SP, p. 1). The new term she coins to denominate her own form of naturalism, "Second Philosophy", makes indirect reference to "first philosophy", the kind of philosophical enterprise Quinean naturalism intended to rule out. "Second Philosophy" aims to be on the side of the scientific-inspired sort of philosophy Quine approved.

For Quine, naturalized philosophy—or Second Philosophy, in Maddy's new terms—must employ *the scientific method*. For Quine, this was a non-problematic recommendation, since for him the scientific method comprised only "observation and the hypothetic-deductive method" (Quine, 1981b, p. 72). However, now that contemporary philosophy of science does not allow any simple definition of scientific method, Maddy has to deal with a difficulty. How to distinguish first philosophy from Second Philosophy? In order to tackle this, she introduces a character, the "Second Philosopher". The Second Philosopher is primarily interested in understanding the world. The description Maddy gives of the Second Philosopher in DA—where she continues to play the central role—is illuminating.

> *Imagine a simple inquirer who sets out to discover what the world is like, the range of what there is and its various properties and behaviors. She begin with her ordinary perceptual beliefs, gradually develops more sophisticated methods of observation and experimentation, of theory construction and testing, and so on […]. She also believes that she and her fellow inquirers are engaged in a highly fallible, but partially and potentially successful exploration of the world, and like anything else, she looks into the matter of how and why the methods she and others use in their inquiries work when they do and don't work when they don't; in these ways, she gradually improves her methods as she goes (Maddy, 2011, DA, p. 38).*

The Second Philosopher is an idealized inquirer who, so to speak, develops the sciences from the scratch. At the beginning of her enterprise, she is an empiricist whose first concern is an understanding of the natural world. As her knowledge grows, new issues arise, such as an inquiry into the reliability of her own methods of investigation. At this stage, typical philosophical questions enter her field of interests. In this sense, philosophy is secondary, that is, it follows after science. Second Philosophical issues arise only inside science, motivated by the needs of scientific enterprise. In fact, the Second Philosopher does not distinguish second philosophical issues from scientific ones. Both kinds of issue are part of a more general inquiry into the world. She has no demarcation criterion between Second Philosophy and science. Even first philosophy is not completely distinguished from them. A remark that Maddy makes about how the Second Philosopher would assess the Cartesian Method of Doubt exemplifies this point.

> *Recall that our Second Philosopher has no grounds on which to denounce First Philosophy as 'unscientific'. Open-minded at all times, she's willing to entertain Descartes's claim that the Method of Doubt will uncover useful knowledge. If, by her lights, it did generate reliable beliefs, she'd have no*

> *scruple about using it. But if it did, by her lights—that is, by lights we tend to describe as 'scientific'—then we'd also be inclined to describe the Method of Doubt as 'scientific' (Maddy, 2007, SP, p. 18, footnote 14).*

The Second Philosopher does not have an ultimate test of scientificity. Some method can be scientific if it proves to be useful in the acquisition of reliable knowledge, but this analysis is made on a case-by-case basis and is relative to the Second Philosopher's lights, which can change over time.

In the same way that the Second Philosopher meets philosophical matters in the course of her investigation of the natural world, she also finds mathematics.

> *[O]ur inquirer begins to notice that logic and arithmetic are essential tools in her efforts to understand the world, and she eventually sees that the calculus, higher analysis, and much of contemporary pure mathematics are also invaluable for getting at the behaviors she studies and for formulating her explanatory theories. This gives her good reason to pursue mathematics herself, as part of her investigation of the world, but she also recognizes that it is developed using methods that appear quite different from the sort of observation, experimentation and theory formation that guide the rest of her research (Maddy, 2011, DA, p. 39).*

At this point, the Second Philosopher realizes that mathematics is, indeed, methodologically autonomous regarding the empirical sciences. It is important to remark that this conclusion is based on a careful examination of mathematical practice, following the same approach taken by the naturalist of NM. The autonomy of mathematics catches her attention and she becomes interested in the methodological peculiarities of mathematics. Again, typical philosophical issues arise.

> *This raises questions of two general types. First, as part of her continual evaluation and assessment of her methods of investigation, she will want an account of the methods of pure mathematics; she will want to know how best to carry on this particular type of inquire. Second, as part of her general study of human practices, she will want an account of what pure mathematics is: what sort of activity is it? what is the nature of its subject matter? how and why does it intertwine so remarkably with her empirical investigations? (Maddy, 2011, DA, p. 39).*

Since these philosophical questions about mathematics arise within the practice of mathematics itself, it can also be said that philosophy is "secondary" to mathematics. Similar to the relationship between science and philosophy, in this case the Second Philosopher does not have a demarcation criterion between mathematics and philosophy either, be it 'first' or 'second.' This may sound confusing, given that in NM Maddy seems to rely on such a distinction when she states, e.g., that philosophical issues do not affect mathematical debates, as we have seen above. Maddy clarifies this point.

> *There [in NM] I phrase the conclusion as recommending 'mathematical' as opposed to 'philosophical' considerations, an unfortunate terminological choice that led to unproductive debate over what counts as 'philosophy' (e.g., is the goal of deriving classical mathematics a 'philosophical' one?). The Second Philosopher—bless her!—doesn't talk this way; just as she employs no demarcation criteria for science vs. non-science, she has no litmus test for philosophy vs. non-philosophy. Instead, she notes […] that considerations of existence and truth and knowledge, of ontology and epistemology, do not in fact play an instrumental role in settling questions of mathematical method. Just as we describe her methods as 'scientific', we might describe those considerations as 'philosophical', but these are just our rough-and-ready way of describing the Second Philosopher's deliberations (Maddy, 2007, SP, p. 349, footnote 12).*

The Second Philosopher is open-minded. She neither refuses nor accepts any consideration based on a firm conception of what counts as 'philosophy,' 'mathematics,' or 'science.' In fact, she does not need such demarcation criteria. She assesses each proposal or piece of belief individually, "'by her lights,' as in the case of the method of doubt mentioned above. As her beliefs themselves are always open to revision, her acceptance or refusal of a certain method or piece of belief can change over time, following the growth of her general understanding of the world. Thus, if she now regards "considerations of existence and knowledge" as indifferent to mathematical debate, this is due to the fact that, given our current mathematical practices, this is so. Her conclusion is not based on an a priori, definitive, and immutable criterion. The Second Philosopher retains the naturalist spirit, which states that the facts of the practice must precede any theory about them.

The Second Philosopher's attitude towards changes in goals in SP

Second Philosophy is not so different from the naturalism of NM. The crucial points of the latter are preserved in Second Philosophy. Mathematics is still regarded as an autonomous enterprise, and the mathematical practice still deserves the protagonism it received in NM. Therefore, the

Second Philosopher's attitude towards changes in the goals of mathematics is unsurprisingly similar to that of NM.

Replacing the "wild example" of NM, in SP Maddy considers a hypothetical situation in which mathematicians have given up the goal of providing tools for the natural sciences, and have engaged themselves in a new enterprise, completely disconnected from the sciences. Although contemporary mathematics, according to Maddy, is not guided by the necessities of the sciences, it cannot be seen as completely disconnected from them. The interest that the Second Philosopher has in mathematics is essentially linked to the fact that the sciences employ mathematics extensively. Moreover, the Second Philosopher understands that the autonomous pursuit of pure mathematics is the best way to provide the sciences with a "well-stocked warehouse" of mathematical resources that can be applied as scientific necessities arise (cf. Maddy, 2007, SP, p. 347). Therefore, even if the methods of pure mathematics are not similar to those of the sciences—such as observation and experimentation—, they play an important role in the broader enterprise of understanding the world. In the hypothetical situation Maddy conceives, the pure mathematicians have forsaken the goal of providing the sciences with a well-stocked warehouse of mathematical resources.

> *[W]hat if mathematicians were to decide that the goal of providing tools for natural science should be outweighed by some other worthy objective, whatever that might be? What if the entire community were to wander off in pursuit of this new goal, leaving science bereft? One unhappy thought seems to me unavoidable: if the Second Philosopher couldn't somehow persuade these hypothetical mathematicians in terms of other shared goals and values—from among those currently in play—she would have no extraordinary means by which to convince them that they were wrong. […] [S]he can show by her mathematical methods why these hypothetical wayward mathematicians are wrong, but she cannot do so, as would be required to return them to the fold, without appeal to the very methods they have forsaken (Maddy, 2007, SP, p. 350).*

Similarly to the naturalist reaction in NM, here Maddy recognizes that there is little to be done concerning the wayward mathematical community. Due to the fact that they have rejected the primary goals, methods, and values of what we know as mathematics, there is no common ground from which the Second Philosopher could convince them to go back to their previous goals.

However, differently from her opinion in NM, in this case the Second Philosopher would not feel constrained to accept the new mathematical methods and goals as legitimate, and nor to take this new state of affairs as a refutation of her former philosophical account of mathematics. The Second Philosopher agrees that, if a philosophical account of mathematics comes into conflict with mathematical practice, it is the philosophy that must give. But in this case there is another issue. Can the new practice still be called mathematics?

> *There's nothing in the strange tale told so far to determine whether or not the practice of these wayward souls would continue to be called 'mathematics', and of course the word doesn't matter. What is clear is that the new practice, whatever it's called, wouldn't play the same role in the Second Philosopher's investigation of the world as the discipline we call 'mathematics' now plays (Maddy, 2007, SP, p. 350-351).*

As a result, the Second Philosopher's attitude regarding this new enterprise would be one of disinterest. Nonetheless, given that the scientific need for mathematical tools would certainly continue unshaken, a practice equivalent to that which we now call mathematics would be invented, even if under a new name. "If the word 'mathematics' were retained by the wayward practice, she [the Second Philosopher] would need a new one, but again, the word isn't what's at issue" (Maddy, 2007, SP, p. 351).

A fundamental point here is that the Second Philosopher does not have a definition of what could count as science or mathematics. "In neither case does she have access to a show-stopper of the form 'x is science iff ...' or 'y is mathematics iff ...'" (Maddy, 2007, SP, p. 350, footnote 17). As a result, she cannot accuse the wayward mathematics of being inauthentic. However, it is important to remark that, although she has sociological-like methods of investigation in her repertoire, Maddy does not subscribe to any sociological definition of science or mathematics, such as "x is science iff the scientists regard x as science" or "y is mathematics iff the mathematicians regard y as mathematics", as we have seen. This allows her to engage in a new enterprise that would play the role that formerly belonged to mathematics. The naturalist did not have a clear view concerning demarcation criteria. Because of this, she had no choice but to accept the new direction that mathematicians took. The Second Philosopher is in a better position regarding this matter.

Even so, Feferman's accusation of transitoriness remains valid regarding Second Philosophy, since the Second Philosopher still cannot establish once and for all a test of appropriateness for mathematical goals. Retaining her former opinion, in SP Maddy regards this transitoriness as positive. She recognizes that even her own philosophical theses are not valid throughout the history of mathematics. The autonomy of mathematics is a recent fact. Maddy quotes Kline (1972) to explain the interdependence between mathematics and sciences in the seventeenth century:

> *[A]s science began to rely more and more upon mathematics to produce its physical conclusions, mathematics began to rely*

> *more and more upon scientific results to justify its own procedures. The upshot of this interdependence was a virtual fusion of mathematics and vast areas of science (Kline, 1972,* in *Maddy, 2007, p. 345).*

In the seventeenth century, a philosophy of mathematics that subordinated mathematics to the sciences, such as Quine does, would have been acceptable.

> *A Second Philosopher in an earlier era would have had an easier time with these questions [regarding the proper methods of mathematics], because … mathematics and natural science weren't always as sharply distinguished as they are today. … For a Second Philosopher in this climate, the study of mathematical methods would be an inseparable part of her general investigation of scientific methods (Maddy, 2007, SP, p. 344-345).*

Therefore, some of the Second Philosopher's contemporary claims about mathematics would have been wrong in the eighteenth century. This is completely natural, as mathematics has changed since then—not only mathematical methods, but also the goals of mathematics. In the eighteenth century, the development of mathematics was guided by the necessities of the sciences. Thus, at that time one could say that mathematics, as well as the sciences, aimed at understanding the natural world. Today, pure mathematics pursues its own goals, not connected with those of the sciences, such that a different philosophical account of mathematics is needed.

In fact, an account of this "historical reversal" in mathematical ends is the main theme of the first chapter of DA. There, Maddy traces the source of these changes to both the mathematics and the sciences of the nineteenth century. From the scientific side, there was a better understanding of how mathematics is applied in the sciences at this point. Far away from the Galilean and Newtonian view that the use of mathematics in scientific theories was literal, the contemporary view regards many mathematical applications in the sciences as idealizations (cf. Maddy, 2011, DA, p. 8-27). From the mathematical side, Maddy (2011, DA, p. 27) remarks that a shift in the interests of mathematicians played a central role: "we've seen how the study of many pure mathematical concepts, structures and theories arose simply because mathematicians began to pursue a range of peculiarly mathematical goals with no immediate connection to applications".

The assessment of the benefit of this change, however, is not straightforward. An analysis of the historical register can, at most, show that changes in mathematical ends really happened. But this does not help in the judgment of its merit. The adoption of conventional means–ends reasoning allows Maddy to evaluate the success of methods with regard to the intended goals, as we have noted above. However, until SP she had not explicitly formulated a parameter to assess the goals themselves.

The Second Philosopher's attitude towards changes in goals in DA

The main innovation in DA is not in the philosophical background. Maddy's preface to DA makes clear its connection to her two previous works. In the first paragraph, she remembers the naturalism of NM and its defence of the autonomy of mathematics:

> *[J]ust as a fundamentally naturalistic perspective counts against* criticizing *a bit of mathematics on the basis of extra-mathematical considerations, it counts just as heavily against* supporting *a bit of mathematics on the basis of extra-mathematical considerations (Maddy, 2011, DA, p. ix).*

Regarding the connection between SP and DA, Maddy says that

> *Second Philosophy ([2007]) lays out the broader philosophical background that seemed to me necessary for a return to those traditional questions [on how to evaluate set-theoretic axioms]; IV.4 of that book contains a brief sketch of what the resulting answers might look like. The goal of the current book, then, is to fill in and develop those sketchy answers (Maddy, 2011, DA, p. ix).*

The last sentence makes clear that Maddy intends to retain the philosophical background of SP. The Second Philosopher is again the protagonist here.

Although her philosophical background is the same, the Second Philosopher's account of mathematical practice has changed substantially. In DA, Maddy claims to have finally found the grounds from which to assess the goals of mathematics. "Mathematical depth" provides these grounds.

The Second Philosopher's *modus operandi* in DA is the same as that employed in SP, that is, analysis of real mathematical practice, both in the history of set theory and in the contemporary debate.[2] From this analysis, she concludes

[2] In DA, Maddy is mainly concerned with set theory and its axioms. There is no explicit intention to generalize her analysis to the whole of mathematics, although this is undertaken in some situations, such as in her explanation of objectivity in mathematics through the concept of mathematical depth. This generalization is not improper, given the foundational role of set theory.

that the most important decisions of the set-theorists were guided by the goal of pursuing what mathematicians refer to using a variety of names, such as "mathematical depth", "mathematical fruitfulness", "mathematical effectiveness", and so on. Roughly, a definition, axiom, or theory is said to be "deep", "fertile", or "effective" if it produces desirable consequences. Zermelo's defence of the Axiom of Choice is a good example. According to Maddy (2011, DA, p. 47), the most decisive point of Zermelo's case in favour of the Axiom of Choice is his claim that the latter is "necessary for science [mathematics]" (Zermelo, 1967, p. 187). In order to show this, Zermelo presents a number of elementary mathematical results that cannot be proven unless the Axiom of Choice is assumed. Without it, mathematics would become a "mutilated science" (Zermelo, 1967, p. 189). Zermelo concludes that the Axiom of Choice must be accepted because of its deep, productive consequences.

Broadly, Maddy claims that the axioms of set theory were generally assumed on a similar basis. The axioms of standard set theory are those that are currently accepted, according to Maddy, primarily because they produce a good theory, in the sense that they allow us to prove most of the central theorems of mathematics at the same time as opening new avenues for research. "These favored candidates [for axiom] differ from alternatives and near-neighbors in that they track what we might call the topography of mathematical depth" (Maddy, 2011, DA, p. 80).

In conformity with the autonomy of mathematics, Maddy points out that mathematical depth is a genuine mathematical goal. When the set-theorist pursues mathematical depth, she is not answering to any extra-mathematical purpose. Rather, she is engaged in an entirely mathematical undertaking whose aim is also mathematical, namely, the production of a fertile theory. Moreover, mathematical depth is an objective feature. The following excerpt explains this aspect.

> *This topography [of mathematical depth] stands over and above the merely logical connections between statements, and furthermore, it is entirely objective: just as it's not up to us which bits of pure mathematics best serve the needs of natural science, just as it's not up to us that it would be counterproductive to insist that all 'groups' be commutative, it's also not up to us that appealing to sets and transfinite ordinals allows us to capture facts about the uniqueness of trigonometric representations, that the Axiom of Choice takes an amazing range of different forms and plays a fundamental role in many different areas, that large cardinals arrange themselves into a hierarchy that serves as an effective measure of consistency strength, that determinacy is the root regularity property for projective sets and interrelates with large cardinals, and so on. These are the facts […] that constrain our set-theoretic methods, and these facts […] are not traceable to ourselves as subjects (Maddy, 2011, DA, p. 80-81).*

Given that the "topography of mathematical depth" is objective, mathematical debates on, for example, the best definition of a concept, or the best axioms to be assumed in a theory, can be objectively settled through an assessment of the extent to which they track mathematical depth. Axioms and definitions that are mathematically deep—that is, that capture features of the topography of mathematical depth—tend to be incorporated to mathematics. This is not a matter of opinion, but a matter of fact.

> *[J]udgments of mathematical depth are not subjective: I might be fond of a certain sort of mathematical theorem, but my idiosyncratic preference doesn't make some conceptual means towards that goal into deep or fruitful or effective mathematics; for that matter, the entire mathematical community could be blind to the virtues of a certain method or enamored of a merely fashionable pursuit without changing the underlying facts of which is and which isn't mathematically important (Maddy, 2011, DA, p. 81).*

Here we start to notice how mathematical depth can take part in the assessment of the appropriateness of goals. Maddy claims that even if a mathematician or a group of mathematicians use the most effective methods in order to pursue a certain goal, this will not be regarded as deep, fruitful, or effective mathematics unless the goal itself is linked to the tracking of the topography of mathematical depth. For example, the rejection of the maxim against inconsistency to benefit the self-esteem of school children—her example in NM, mentioned above—could hardly be seen as producing deep mathematics or a fertile theory. Thus, this would not be an appropriate goal.

> *The key here is that mathematical fruitfulness isn't defined as 'that which allows us to meet our goals', irrespective of what these might be; rather, our mathematical goals are only proper insofar as satisfying them furthers our grasp of the underlying strains of mathematical fruitfulness. In other words, the goals are answerable to the facts of mathematical depth, not the other way 'round. Our interests will influence which areas of mathematics we find most attractive or compelling, just as our interest influence which parts of natural science we're most eager to pursue, but no amount of partiality or neglect from us can make a line of mathematics fruitful if it isn't, or fruitless if it is (Maddy, 2011, DA, p. 82).*

And, in a footnote to this point:

> *Here at last are grounds on which to reject the nihilism of footnote 9 on p. 198 of [1997], and even the tempered version in [2007], pp. 350-351. If mathematicians wander off the path of mathematical depth, they're going astray, even if no one realizes it (Maddy, 2011, DA, p. 82, footnote 42).*

Footnote 9 on p. 198 of NM is quoted above. The "tempered version" of SP is discussed in the previous section. In the excerpt quoted above, Maddy remarks that her position in DA concerning changes in the goals of mathematics is completely different from those of NM and SP. In DA, Maddy can finally answer Feferman's question "as to what determines the 'mathematical ends' for which the 'most effective' means are to be sought" (Feferman *et al.*, 2000, p. 409). Essentially, it is the concept of mathematical depth that, in DA, provides the "independent grounds on which to criticize or defend the actual goals of the practice" (Maddy, 1997, NM, p. 198), which are denied to the naturalist in NM and to the Second Philosopher in SP. However, Maddy's philosophical background has not explicitly changed, at least since SP. Consequently, we can ask whether some crucial traits of her philosophy, such as the autonomy of mathematics, the absence of a litmus test for mathematicity, and the acknowledged positive transitoriness of her theses are still valid. I suspect that these keystones of her philosophy are not compatible with the existence of an ultimate criterion for the evaluation of the ends of mathematics.

The most obvious problem is related to the definition of mathematics. In SP, the Second Philosopher states that she does not have a definition of the form 'y is mathematics iff ...' (Maddy, 2007, SP, p. 350, footnote 17). The Second Philosopher regards this as a positive characteristic, as we explained above, due to her open-mindedness. However, once the concept of mathematical depth is instituted as the final goal of mathematics, a definition of mathematics follows immediately: y is mathematics iff y is an undertaking that furthers our grasp of the underlying strains of mathematical fruitfulness. This would be beneficial, if it did not jeopardize the autonomy of mathematics and the declared positive transitoriness of Maddy's philosophy of mathematics.

The autonomy of mathematics and the transitory character of Maddy's philosophy seem to be inevitably intertwined in her previous works. Since mathematics is autonomous, and given that history shows that mathematical methods and goals change over time, any philosophical account of the methodology of mathematics that aims to be primarily descriptive, rather than normative, must be transitory. It is worth emphasizing that a philosophy that recognizes the autonomy of mathematics has no choice but to limit itself to being descriptive. If mathematics is autonomous, only mathematics can prescribe norms to itself. Furthermore, Maddy supported the opinion that transitoriness was positive because philosophical theses must change in order to catch up with the progress of scientific and mathematical knowledge.

In DA, however, as we have seen, Maddy institutes a limit to philosophically admissible changes in mathematics: whatever goal the mathematicians chose to pursue, it will be regarded as adequate only if mathematical depth is preserved as the focus of research. Now that there is such a parameter of adequacy for mathematical goals, Maddy's philosophy apparently becomes normative. However, we cannot straightforwardly conclude that this breaks the autonomy of mathematics. As the Second Philosopher does not have clear demarcation criteria for distinguishing between mathematics and philosophy, we must first consider whether the concept of mathematical depth is genuinely mathematical or not.

In order to accomplish this task, it is worth comparing Maddy's account of mathematical depth in DA with her account of maximization in NM. Can the concept of mathematical depth be defined in set theory, like maximization? Probably not, at least in the way it is developed in DA. Maddy herself acknowledges that her account of mathematical depth "remains uncomfortably metaphorical" (Maddy, 2011, DA, p. 117). Besides this, we can add that the concept of mathematical depth can perfectly integrate a philosophical or scientific-historical-sociological account of mathematics, but it is not likely to integrate any mathematical theory. Undoubtedly the examples of mathematical depth Maddy mentions are genuine mathematical ones. As she emphasizes, in these episodes mathematical considerations independent of philosophical opinions settled the debate. Nonetheless, what has little to do with mathematics is the claim that these cases are examples of situations where mathematicians tracked the "topography of mathematical depth" (Maddy, 2011, DA, p. 80). This claim is as extra-mathematical as a platonist claim would be that affirmed that these cases are examples of situations where mathematicians traced the nature of an independent reality of mathematical entities. This is easy to see, if, in a Carnapian way (cf. Carnap, 2005), we imagine two hypothetically equally competent mathematicians with divergent philosophical opinions, both investigating the properties of certain sets. The first one, convinced of Maddy's explanation, believes that his findings trace the topography of mathematical depth. In contrast, the second one, a Platonist, believes that he is discovering characteristics of independent entities. If we assume Maddy's claim about the insignificance of philosophical positions in mathematical debates, we can suppose that, despite their different opinions, regarding genuine mathematical questions both mathematicians would agree. If they disagree, however, they would probably agree that discussion of the mathematical details of their arguments would be more effective in settling the issue than a philosophical debate about their divergent views. Therefore, remembering the distinction Maddy makes between *prose* and *calculus* (Maddy, 1997, NM, p. 166 ff), we can say that the concept of mathematical depth is *prose*, and not *calculus*.

In an excerpt quoted above, Maddy affirms that "considerations of existence and truth and knowledge, of ontology and epistemology, do not in fact play an instrumental role in settling questions of mathematical method" (Maddy, 2007, SP, p. 349, footnote 12). Paraphrasing Maddy, we can say that considerations of mathematical depth—such as "this tracks the topography of mathematical depth"—, and likewise Platonist considerations—such as "this tracks the proprieties of independent entities"—, do not in fact play a role in settling questions of mathematical method. Thus, in a rough-and-ready way, we might describe those considerations as 'philosophical.' Therefore, the norm "mathematical goals are only proper insofar as satisfying them furthers our grasp of the underlying strains of mathematical fruitfulness" (Maddy, 2011, DA, p. 82) qualifies as extra-mathematical, in the same way that a norm such as "mathematical goals are only proper insofar satisfying them furthers our grasp of the underlying independent reality of sets" so qualifies. Consequently, this is incompatible with the alleged autonomy of mathematics.

When "wild examples" are considered, such as the admission of inconsistency to benefit of school children, a certain degree of limitation on the autonomy of mathematics might sound more salutary than detrimental. Nevertheless, is confinement to the pursuit of mathematical depth a salutary limitation? Let us suppose a situation in which the new goals are not so weird. For example, we can imagine a hypothetical future where innovative scientific research increasingly demands unusual new applications of mathematics. In this future, the mathematical community would perhaps develop a renewed interest in applied mathematics, and this might end up stagnating areas, methods, and ways of reasoning related to pure mathematics. In such a hypothetical future scenario, perhaps the most appropriate methods to settle debates in mathematics would again be related to applicability in the natural sciences, and not to mathematical depth. In this future, the naturalist maxim—if a philosophical account of mathematics comes into conflict with mathematical practice, it is the philosophy that must give—, which is apparently endorsed by Second Philosophy, would seem more judicious than adherence to an old-fashioned goal.

Furthermore, it is worth remembering that one of the objectives of the naturalist in NM is to contribute to mathematical debate. The Second Philosopher apparently preserves this objective. However, a philosophy that sticks to mathematical depth would be as irrelevant to mathematical debate in this hypothetical future in the same way that a philosophy that sticks to arguments about the applicability of mathematics in the natural sciences is irrelevant to the contemporary debate, if Maddy's analysis of current mathematical practice is correct. If the Second Philosopher aims at contributing to mathematical debate, she should follow changes in mathematical debate, or she will have little chance to thrive.

Another issue affected by the institution of mathematical depth as the ultimate goal of mathematics concerns Maddy's criticism of Quine's indispensability argument. How can Maddy legitimately refuse to subordinate mathematics to science? For Maddy admits, as we have seen, that there was an age when mathematical research was subordinated to scientific goals. What she does not agree with is continuing to subordinate contemporary mathematics to the empirical sciences, because mathematics has changed since then. Quine also recognizes that mathematics has changed, but he regards some parts of pure mathematics as "mathematical recreation" or, so to speak, as mathematics that has gone astray. According to Quine, we should stay close to applied mathematics; according to Maddy, we should stay close to mathematical depth. How to decide between them? How can we not conclude that, in reality, it was pure mathematics that went astray? Maddy criticizes Quine's position based on contemporary mathematical practice, but she does not explain why contemporary practice must be privileged to the detriment of, say, the practice of the seventeenth century. As we have seen, in DA Maddy admits that the mathematical community as a whole could go astray, even if nobody realizes it. Therefore, how to know whether the whole community has not already lost itself in the past, when it started to prioritize pure mathematics over application? If mathematicians have already gone astray, Quine would be right.

At the beginning of this section I quoted the preface to DA, in which Maddy claims that extra-mathematical considerations can neither criticize nor support mathematics. Is Maddy betraying this basic principle of her naturalism? First, if the concept of mathematical depth is really not genuinely mathematical, as I concluded above, then it seems that the institution of mathematical depth as the ultimate goal of mathematics philosophically *supports* contemporary pure mathematics. Moreover, even if Maddy is not criticizing current mathematics in DA, the institution of mathematical depth as the permanent goal of mathematics opens the door to future criticism of mathematics. The point is: for someone who assumes that mathematics is autonomous and, at the same time, recognizes that it changes over time, it is not possible to institute the goal of a particular age as the permanent goal of mathematics.

Despite the difficulties I have pointed out above, if Maddy's conclusions are taken as simply a philosophical or historical-sociological explanation of the phenomenon of objectivity in contemporary mathematics, the idea that mathematical research aims to track mathematical depth has its merits. Her "post-metaphysical" account of objectivity in mathematics, as she calls it (cf. Maddy, 2011, DA, p. 116), has the potential to exchange the mysterious world of abstract entities of traditional Platonist explanations for a set of characteristics under the name of "mathematical depth" that mathematicians can recognize using their conventional methods. Furthermore, simply explaining the methodology of mathematics does not threaten the autonomy of mathematics. If Maddy's explanation is understood as a transitory explanation—true regarding contemporary mathematics, but not forever—it is compatible with the idea that mathematics can carry on tracking mathematical depth or whatever else by its own lights, or can change its goals according to its own development. The problem is not the concept of mathematical depth itself, but Maddy's affirmation that, from this idea, she has the grounds "on which to reject the nihilism" of naturalism in Mathematics. In doing so, I suspect that Maddy rejects crucial traits of both her naturalism and Second Philosophy.

As can be perceived from the quotations from DA above, Maddy does not linger on explaining why she modified her attitude towards changes in goals in mathematics. She gives little more than a footnote to this issue. Considering this lack of emphasis, perhaps we should not take this change as seriously as we have. On the other hand, although it is only quickly mentioned in DA, the change can be seen as the result of a characteristic that has been present in Maddy's works since at least NM. There is a tension in Maddy's philosophy between prioritizing mathematical practice—she takes the methodological consensus of the mathematical community as a parameter to evaluate any philosophy of mathematics—and rejecting the idea that mathematics is just what mathematicians do. The institution of mathematical depth as the ultimate goal of mathematics can be seen as a result of this tension. Mathematical depth is the decisive factor, she claims, and mathematical practice is nothing more than a consequence of mathematical depth. Mathematicians do what they do because they are tracing mathematical depth. However, after the above discussion, we can ask if Maddy's naturalism is compatible with any philosophical position that proposes something other than mathematical practice itself as the decisive factor. Would the methodological consensus of the mathematical community and its transformations over time be the decisive factors that autonomously shape and set the goals of mathematical research? Maddy flirts with a historical-sociological conception of mathematics such as this, but rejects it systematically. The concept of mathematical depth is her final move, at least for now, to distance herself from a historical–sociological conception of mathematics. However, this endangers her naturalism and Second Philosophy, as I have sustained here.

The crux of the problem is that the institution of a permanent parameter for judging mathematical ends has the potential to convert Second Philosophy into first philosophy. When the Second Philosopher ventures to take this concept as a vantage point from which to judge mathematics, she is instituting an extra-mathematical tribunal. But Maddy had vetoed this movement twice in the past. Provided that Maddy's philosophical background in DA is the same as in SP, namely, Second Philosophy, and that Second Philosophy is not far from her former naturalism, I believe that this movement is incompatible with her philosophical commitments. The problem is not the idea of mathematical depth itself, nor its employment in the explanation of methodological decisions in contemporary pure mathematics. What really threatens the Second Philosophical spirit, in my view, is the conversion of the explanation into a rule, into a parameter by which to judge mathematics itself. Can Second Philosophy be normative? This question is complex, but I admit that there is some margin for normativeness in Second Philosophy. For example, a kind of normativeness based on "hypothetical imperatives", such as that which Laudan (1987) proposes, is perfectly suitable to the Second Philosopher, since she resorts to conventional means–ends reasoning to assess methodological decisions in mathematics. Thus, based on this standard, she can state hypothetical imperatives of the form "if mathematicians' goal is *y*, then they ought to do *x*". In fact, Maddy acknowledges this normative role of her naturalism when she states her intention to contribute to contemporary mathematical debates. Moreover, she remarks that "such instrumental [means-ends] reasoning is not external to mathematics; it is used in mathematics as it is in all human undertakings" (Maddy, 1997, NM, p. 197). Therefore, hypothetical imperatives of this sort could not offend the autonomy of mathematics. Nevertheless, when it comes to the assessment of the goals themselves, it is difficult to conceive of a way of establishing a permanent parameter without offending the autonomy of mathematics—or, at least, the autonomy of a possible future mathematics. It seems that the concession of autonomy to mathematics, combined with the recognition that mathematical ends change throughout the course of history, prevents the Second Philosopher from being normative in this issue. If this is so, the Second Philosopher has no choice but to accept the nihilism of NM and SP.

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Submitted on January 24, 2016
Accepted on October 05, 2016